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2005 J. Phys. A: Math. Gen. 38 6147

(http://iopscience.iop.org/0305-4470/38/27/005)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 6147–6166 doi:10.1088/0305-4470/38/27/005

Eigenvalues of $\mathcal{PT}$-symmetric oscillators with polynomial potentials

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Received 21 February 2005
Published 22 June 2005
Online at stacks.iop.org/JPhysA/38/6147

Abstract

We study the eigenvalue problem $-u''(z) - [(\mathrm{i}z)^m + P_{m-1}(\mathrm{i}z)]u(z) = \lambda u(z)$ with the boundary condition that $u(z)$ decays to zero as z tends to infinity along the rays $\arg z = -\frac{\pi}{2} \pm \frac{2\pi}{m+2}$ in the complex plane, where $P_{m-1}(z) = a_1 z^{m-1} + a_2 z^{m-2} + \cdots + a_{m-1} z$ is a polynomial and integers $m \geqslant 3$. We provide an asymptotic expansion of the eigenvalues λ_n as $n \to +\infty$, and prove that for each *real* polynomial P_{m-1}, the eigenvalues are all real and positive, with only finitely many exceptions.

PACS numbers: 02.30.Hq, 02.30.Mv, 03.65.Ge, 02.70.Hm
Mathematics Subject Classification: 34L40, 34L20

1. Introduction

For integers $m \geqslant 3$ fixed, we are considering the 'non-standard' non-self-adjoint eigenvalue problems

$$Hu(z,\lambda) := \left[-\frac{\mathrm{d}^2}{\mathrm{d}z^2} - (\mathrm{i}z)^m - P_{m-1}(\mathrm{i}z)\right]u(z,\lambda) = \lambda u(z,\lambda), \qquad \text{for some } \lambda \in \mathbb{C}, \tag{1}$$

with the boundary condition that

$$u(z,\lambda) \to 0 \text{ exponentially, as } z \to \infty \text{ along the two rays } \arg(z) = -\frac{\pi}{2} \pm \frac{2\pi}{m+2}, \tag{2}$$

where P_{m-1} is a polynomial of degree at most $m-1$ of the form

$$P_{m-1}(z) = a_1 z^{m-1} + a_2 z^{m-2} + \cdots + a_{m-1} z, \quad a_j \in \mathbb{C} \qquad \text{for} \quad 1 \leqslant j \leqslant m-1. \tag{3}$$

We let

$$a := (a_1, a_2, \ldots, a_{m-1}) \in \mathbb{C}^{m-1}$$

be the coefficient vector of $P_{m-1}(z)$. We are mainly interested in the case when P_{m-1} is real, that is, when $a \in \mathbb{R}^{m-1}$. However, some interesting facts in this paper hold also for $a \in \mathbb{C}^{m-1}$. So except for theorem 4 below, we will allow $a \in \mathbb{C}^{m-1}$.

If a nonconstant function u satisfies (1) with some $\lambda \in \mathbb{C}$ and the boundary condition (2), then we call λ an *eigenvalue* of H and u an *eigenfunction of H associated with the eigenvalue* λ. Also, the *geometric multiplicity of an eigenvalue* λ is the number of linearly independent eigenfunctions associated with the eigenvalue λ. The operator H in (1) with potential $V(z) = -(\mathrm{i}z)^m - P_{m-1}(\mathrm{i}z)$ is called $\mathcal{PT}$*-symmetric* if $\overline{V(-\overline{z})} = V(z)$, $z \in \mathbb{C}$. Note that $V(z) = -(\mathrm{i}z)^m - P_{m-1}(\mathrm{i}z)$ is a $\mathcal{PT}$-symmetric potential if and only if $a \in \mathbb{R}^{m-1}$.

Before we state our main theorems, we first introduce some known facts by Sibuya [19] about the eigenvalues λ_n of H, numbered in the order of nondecreasing magnitudes.

Theorem 1. *The eigenvalues λ_n of H have the following properties.*

(I) The set of all eigenvalues is a discrete set in $\mathbb{C}$.
(II) The geometric multiplicity of every eigenvalue is 1.
(III) Infinitely many eigenvalues, accumulating at infinity, exist.
(IV) The eigenvalues have the following asymptotic expansion:

$$\lambda_n = \left(\frac{\Gamma\left(\frac{3}{2}+\frac{1}{m}\right)\sqrt{\pi}\left(n-\frac{1}{2}\right)}{\sin\left(\frac{\pi}{m}\right)\Gamma\left(1+\frac{1}{m}\right)}\right)^{\frac{2m}{m+2}}[1+o(1)], \quad \textit{as } n \textit{ tends to infinity}, \quad n \in \mathbb{N}, \tag{4}$$

where the error term $o(1)$ could be complex-valued.

This paper is organized as follows. In section 2, we will introduce work of Hille [13] and Sibuya [19], regarding properties of solutions of (1). We then improve on the asymptotics of a certain function in [19]. In section 3, we introduce an entire function $C(a, \lambda)$ whose zeros are the eigenvalues of H, due to Sibuya [19]. In section 4, we then provide asymptotics of $C(a, \lambda)$ as $\lambda \to \infty$ in the complex plane, improving the asymptotics of $C(a, \lambda)$ in [19]. In section 5, we will improve the asymptotic expansion (4) of the eigenvalues. In particular, we will prove the following. Throughout this paper, we use that $\lfloor x \rfloor$ is the largest integer that is less than or equal to $x \in \mathbb{R}$.

Theorem 2. *Let $a \in \mathbb{C}^{m-1}$ be fixed. Then, there exist $e_\ell(a) \in \mathbb{C}$, $1 \leqslant \ell \leqslant \frac{m}{2}+1$ such that the eigenvalues λ_n of H have the asymptotic expansion*

$$\lambda_n \underset{n\to+\infty}{=} \lambda_{0,n} + \sum_{\ell=1}^{\lfloor \frac{m}{2}+1 \rfloor} e_\ell(a)\lambda_{0,n}^{1-\frac{\ell}{m}} + o\left(\lambda_{0,n}^{\frac{1}{2}-\frac{1}{m}}\right), \tag{5}$$

where

$$\lambda_{0,n} = \left(\frac{\left(n+\frac{1}{2}\right)\pi}{K_m \sin\left(\frac{2\pi}{m}\right)}\right)^{\frac{2m}{m+2}} \quad \textit{with} \quad K_m = \int_0^\infty \left(\sqrt{1+t^m} - \sqrt{t^m}\right) \mathrm{d}t > 0.$$

One can compute K_m directly (or see equation (2.22) in [10] with the identity $\Gamma(s)\Gamma(1-s) = \pi \csc(\pi s)$) and obtains

$$K_m = \frac{\sqrt{\pi}\Gamma\left(1+\frac{1}{m}\right)}{2\cos\left(\frac{\pi}{m}\right)\Gamma\left(\frac{3}{2}+\frac{1}{m}\right)}.$$

In the last section, we prove the following theorem, regarding monotonicity of $|\lambda_n|$.

Theorem 3. *For each $a \in \mathbb{C}^{m-1}$, there exists $M > 0$ such that $|\lambda_n| < |\lambda_{n+1}|$ if $n \geqslant M$.*

This is a consequence of (5).

Finally, when H is $\mathcal{PT}$-symmetric (i.e., $a \in \mathbb{R}^{m-1}$), $u(z, \lambda)$ is an eigenfunction associated with an eigenvalue λ if and only if $\overline{u(-\overline{z}, \lambda)}$ is an eigenfunction associated with the eigenvalue

$\overline{\lambda}$. Thus, the eigenvalues either appear in complex conjugate pairs or else are real. So theorem 3 implies the following.

Theorem 4. *Suppose that $a \in \mathbb{R}^{m-1}$. Then, the eigenvalues λ of H are all real and positive, with only finitely many exceptions.*

For the rest of the introduction, we will mention a brief history of problem (1).

In recent years, these $\mathcal{PT}$-symmetric operators have gathered considerable attention, because ample numerical and asymptotic studies suggest that many of such operators have real eigenvalues only even though they are not self-adjoint. In particular, the differential operators H with some polynomial potential V and with the boundary condition (2) have been considered by Bessis and Zinn-Justin [5], Bender and Boettcher [2] and many other physicists [3–6, 10, 14–16, 18, 20].

Around 1992, Bessis and Zinn-Justin [5] conjectured that when $V(z) = \mathrm{i}z^3 + \beta z^2$, $\beta \in \mathbb{R}$, the eigenvalues are all real and positive, and in 1998, Bender and Boettcher [2] conjectured that when $V(z) = -(\mathrm{i}z)^m + \beta z^2$, $\beta \in \mathbb{R}$, the eigenvalues are all real and positive. Many numerical, asymptotic and analytic studies support these conjectures (see, e.g., [3–6, 10, 14–16, 18, 20] and references therein and below).

The first rigorous proof of reality and positivity of the eigenvalues of some non-self-adjoint H in (1) was given by Dorey, Dunning and Tateo in 2001 [9]. They proved that the eigenvalues of H with the potential $V(z) = -(\mathrm{i}z)^{2m} - \alpha(\mathrm{i}z)^{m-1} + \frac{\ell(\ell+1)}{z^2}$, $m, \alpha, \ell \in \mathbb{R}$, are all real if $m > 1$ and $\alpha < m + 1 + |2\ell + 1|$, and positive if $m > 1$ and $\alpha < m + 1 - |2\ell + 1|$.

Then, in 2002, the present author [17] extended the polynomial potential results of Dorey, Dunning and Tateo to more general polynomial cases, by adapting the method in [9]. Namely, when $V(z) = -(\mathrm{i}z)^m - P_{m-1}(\mathrm{i}z)$, the eigenvalues are all real and positive, provided that for some $1 \leqslant j \leqslant \frac{m}{2}$ the coefficients of the real polynomial P_{m-1} satisfy $(j - k)a_k \geqslant 0$ for all $1 \leqslant k \leqslant m - 1$.

However, there are some $\mathcal{PT}$-symmetric polynomial potentials that produce non-real eigenvalues. Delabaere and Pham [7] and Delabaere and Trinh [8] studied the potential $\mathrm{i}z^3 + \gamma \mathrm{i}z$ and showed that a pair of non-real eigenvalues develops for large negative γ. Moreover, Handy [11] and Handy, Khan, Wang and Tymczak [12] showed that the same potential admits a pair of non-real eigenvalues for small negative values of $\gamma \approx -3.0$. Also, Bender, Berry, Meisinger, Savage and Simsek [1] considered the problem with the potential $V(z) = z^4 + \mathrm{i}Az$, $A \in \mathbb{R}$, under decaying boundary conditions at both ends of the real axis, and their numerical study showed that more and more non-real eigenvalues develop as $|A| \to \infty$. So without any further restrictions on the coefficients $a_k \in \mathbb{R}$, theorem 4 is the most general result one can expect about reality of eigenvalues.

Also, the method used to prove theorem 4 in this paper is new. The method used in [9, 17] is useful in proving reality of *all* eigenvalues, but I think that some critical arguments in proving reality of eigenvalues in [9, 17] cannot be applied to the cases when some non-real eigenvalues exist. The asymptotic expansion (5) itself is interesting, and also (5) implies theorem 3. Note that (4) is not enough to conclude theorem 3. Finally, theorem 3 and $\mathcal{PT}$-symmetry of H explained right before theorem 4 above imply the partial reality of the eigenvalues in theorem 4.

2. Properties of the solutions

In this section, we introduce work of Hille [13] and Sibuya [19] about properties of the solutions of (1).

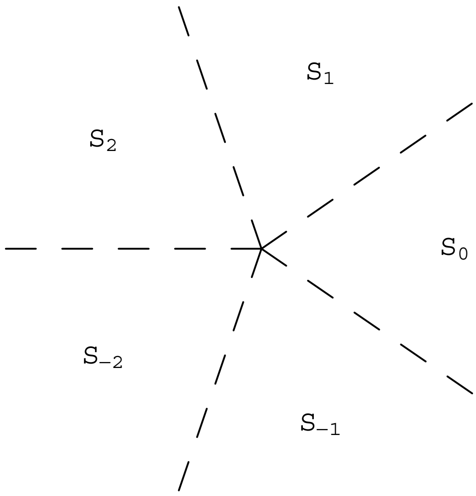

Figure 1. The Stokes sectors for $m = 3$. The dashed rays represent $\arg z = \pm\frac{\pi}{5}, \pm\frac{3\pi}{5}, \pi$.

First, we scale equation (1) because many facts that we need later are stated for the scaled equation. Let u be a solution of (1) and let $v(z,\lambda) = u(-\mathrm{i}z,\lambda)$. Then, v solves

$$-v''(z,\lambda) + [z^m + P_{m-1}(z) + \lambda]v(z,\lambda) = 0, \tag{6}$$

where $m \geqslant 3$ and P_{m-1} is a polynomial (possibly, $P_{m-1} \equiv 0$) of the form (3).

Since we scaled the argument of u, we must rotate the boundary conditions. We state them in a more general context by using the following definition.

Definition. *The Stokes sectors S_k of the equation (6) are*

$$S_k = \left\{ z \in \mathbb{C} : \left| \arg(z) - \frac{2k\pi}{m+2} \right| < \frac{\pi}{m+2} \right\}, \qquad \textit{for} \quad k \in \mathbb{Z}.$$

See figure 1. It is known from Hille [13, section 7.4] that every nonconstant solution of (6) either decays to zero or blows up exponentially, in each Stokes sector S_k. More precisely, one has the following result.

Lemma 5 ([13, section 7.4])**.**

(i) For each $k \in \mathbb{Z}$, every solution v of (6) (with no boundary conditions imposed) is asymptotic to

$$(constant) z^{-\frac{m}{4}} \exp\left[\pm \int^z [\xi^m + P_{m-1}(\xi) + \lambda]^{\frac{1}{2}} \, \mathrm{d}\xi \right] \tag{7}$$

as $z \to \infty$ in every closed subsector of S_k.

(ii) If a nonconstant solution v of (6) decays in S_k, it must blow up in $S_{k-1} \cup S_{k+1}$. However, when v blows up in S_k, v need not be decaying in S_{k-1} or in S_{k+1}.

Lemma 5(i) implies that if v decays along one ray in S_k, then it decays along all rays in S_k. Also, if v blows up along one ray in S_k, then it blows up along all rays in S_k. Thus, since the rotation $z \mapsto \mathrm{i}z$ maps the two rays in (2) onto the centre rays of S_{-1} and S_1,

the boundary conditions on u in (1) mean that v decays in $S_{-1} \cup S_1$.

Next we will introduce Sibuya's results, but first we define a sequence of complex numbers b_j in terms of the a_k and λ, as follows. For $\lambda \in \mathbb{C}$ fixed, we expand

$$\begin{aligned}(1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_{m-1} z^{1-m} + \lambda z^{-m})^{1/2} \\ &= 1 + \sum_{k=1}^{\infty} \binom{\frac{1}{2}}{k} (a_1 z^{-1} + a_2 z^{-2} + \cdots + a_{m-1} z^{1-m} + \lambda z^{-m})^k \\ &= 1 + \sum_{j=1}^{\infty} \frac{b_j(a,\lambda)}{z^j}, \qquad \text{for large} \quad |z|. \end{aligned} \tag{8}$$

Note that $b_1, b_2, \ldots, b_{m-1}$ do not depend on λ, so we write $b_j(a) = b_j(a,\lambda)$ for $j = 1, 2, \ldots, m-1$. So the above expansion without the λz^{-m} term still gives b_j for $1 \leqslant j \leqslant m-1$. We further define $r_m = -\frac{m}{4}$ if m is odd and $r_m = -\frac{m}{4} - b_{\frac{m}{2}+1}(a)$ if m is even.

The following theorem is a special case of theorems 6.1, 7.2, 19.1 and 20.1 of Sibuya [19] that is the main ingredient of the proofs of the main results in this paper.

Theorem 6. *Equation (6), with $a \in \mathbb{C}^{m-1}$, admits a solution $f(z,a,\lambda)$ with the following properties.*

(i) $f(z,a,\lambda)$ is an entire function of z, a and λ.

(ii) $f(z,a,\lambda)$ and $f'(z,a,\lambda) = \frac{\partial}{\partial z} f(z,a,\lambda)$ admit the following asymptotic expansions. Let $\varepsilon > 0$. Then,

$$f(z,a,\lambda) = z^{r_m}(1 + O(z^{-1/2})) \exp[-F(z,a,\lambda)],$$
$$f'(z,a,\lambda) = -z^{r_m + \frac{m}{2}}(1 + O(z^{-1/2})) \exp[-F(z,a,\lambda)],$$

as z tends to infinity in the sector $|\arg z| \leqslant \frac{3\pi}{m+2} - \varepsilon$, uniformly on each compact set of (a,λ)-values. Here,

$$F(z,a,\lambda) = \frac{2}{m+2} z^{\frac{m}{2}+1} + \sum_{1 \leqslant j < \frac{m}{2}+1} \frac{2}{m+2-2j} b_j(a) z^{\frac{1}{2}(m+2-2j)}.$$

(iii) Properties (i) and (ii) uniquely determine the solution $f(z,a,\lambda)$ of (6).

(iv) For each fixed $a \in \mathbb{C}^{m-1}$ and $\delta > 0$, f and f' also admit the asymptotic expansions,

$$f(0,a,\lambda) = [1 + o(1)]\lambda^{-1/4} \exp[L(a,\lambda)], \tag{9}$$

$$f'(0,a,\lambda) = -[1 + o(1)]\lambda^{1/4} \exp[L(a,\lambda)], \tag{10}$$

as $\lambda \to \infty$ in the sector $|\arg(\lambda)| \leqslant \pi - \delta$, where

$$L(a,\lambda) = \begin{cases} \int_0^{+\infty} \left(\sqrt{t^m + P_{m-1}(t) + \lambda} - t^{\frac{m}{2}} - \sum_{j=1}^{\frac{m+1}{2}} b_j(a) t^{\frac{m}{2}-j}\right) \mathrm{d}t, \\ \quad \text{if } m \text{ is odd}, \\ \int_0^{+\infty} \left(\sqrt{t^m + P_{m-1}(t) + \lambda} - t^{\frac{m}{2}} - \sum_{j=1}^{\frac{m}{2}} b_j(a) t^{\frac{m}{2}-j} - \frac{b_{\frac{m}{2}+1}(a)}{t+1}\right) \mathrm{d}t, \\ \quad \text{if } m \text{ is even}. \end{cases}$$

(v) The entire functions $\lambda \mapsto f(0,a,\lambda)$ and $\lambda \mapsto f'(0,a,\lambda)$ have orders $\frac{1}{2} + \frac{1}{m}$.

Proof. In Sibuya's book [19], see theorem 6.1 for a proof of (i) and (ii); theorem 7.2 for a proof of (iii); and theorem 19.1 for a proof of (iv). Moreover, (v) is a consequence of (iv) along with theorem 20.1. Note that properties (i), (ii) and (iv) are summarized on pages 112–3 of Sibuya [19]. □

Using this theorem, Sibuya [19, theorem 19.1] also showed the following corollary that will be useful later on.

Corollary 7. *Let $a \in \mathbb{C}^{m-1}$ be fixed. Then, $L(a, \lambda) = K_m \lambda^{\frac{1}{2}+\frac{1}{m}}(1 + o(1))$ as λ tends to infinity in the sector $|\arg \lambda| \leqslant \pi - \delta$, and hence*

$$\mathrm{Re}\,(L(a, \lambda)) = K_m \cos\left(\frac{m+2}{2m}\arg(\lambda)\right)|\lambda|^{\frac{1}{2}+\frac{1}{m}}(1 + o(1)), \tag{11}$$

as $\lambda \to \infty$ in the sector $|\arg(\lambda)| \leqslant \pi - \delta$.

In particular, $\mathrm{Re}\,(L(a, \lambda)) \to +\infty$ *as $\lambda \to \infty$ in any closed subsector of the sector $|\arg(\lambda)| < \frac{m\pi}{m+2}$. In addition,* $\mathrm{Re}\,(L(a, \lambda)) \to -\infty$ *as $\lambda \to \infty$ in any closed subsector of the sectors $\frac{m\pi}{m+2} < |\arg(\lambda)| < \pi - \delta$.*

Proof. This asymptotic expansion will be clear from lemma 8 below or, alternatively, see [19, theorem 19.1] for a proof. □

Based on the above corollary, Sibuya [19, theorem 29.1] also proved the following asymptotic expansion of the eigenvalues:

$$\lambda_n = \omega^m \left(\frac{(-2n+1)\pi}{2K_m \sin\left(\frac{2\pi}{m}\right)}\right)^{\frac{2m}{m+2}} [1 + o(1)], \qquad \text{as} \quad n \to \infty, \tag{12}$$

where

$$\omega = \exp\left[\frac{2\pi \mathrm{i}}{m+2}\right].$$

Note that in this paper we consider the boundary conditions of the scaled equation (6) where v decays in $S_{-1} \cup S_1$, while Sibuya studies equation (6) with boundary conditions such that v decays in $S_0 \cup S_2$. The factor ω^m in our formula (12) is due to this scaling of the problem.

Remark. Throughout this paper, we will deal with numbers like $(\omega^\nu \lambda)^s$ for some $s \in \mathbb{R}$, and $\nu \in \mathbb{C}$. As usual, we will use

$$\omega^\nu = \exp\left[\nu \frac{2\pi \mathrm{i}}{m+2}\right]$$

and if $\arg(\lambda)$ is specified, then

$$\arg((\omega^\nu \lambda)^s) = s[\arg(\omega^\nu) + \arg(\lambda)] = s\left[\mathrm{Re}(\nu)\frac{2\pi}{m+2} + \arg(\lambda)\right], \qquad s \in \mathbb{R}.$$

If $s \notin \mathbb{Z}$, then the branch of λ^s is chosen to be the negative real axis.

Next, we provide an improved asymptotic expansion of L. We will use this new asymptotic expansion of L to improve the asymptotic expansion (12) of the eigenvalues.

Lemma 8. *Let $m \geqslant 3$ and $a \in \mathbb{C}^{m-1}$ be fixed. Then, there exist constants $K_{m,j}(a) \in \mathbb{C}$, $0 \leqslant j \leqslant \frac{m}{2} + 1$, such that*

$$L(a, \lambda) = \begin{cases} \sum_{j=0}^{\frac{m+1}{2}} K_{m,j}(a)\lambda^{\frac{1}{2}+\frac{1-j}{m}} + O\left(|\lambda|^{-\frac{1}{2m}}\right), & \text{if } m \text{ is odd,} \\ \sum_{j=0}^{\frac{m}{2}+1} K_{m,j}(a)\lambda^{\frac{1}{2}+\frac{1-j}{m}} - \frac{b_{\frac{m}{2}+1}(a)}{m}\ln(\lambda) + O\left(|\lambda|^{-\frac{1}{m}}\right), & \text{if } m \text{ is even,} \end{cases}$$

as $\lambda \to \infty$ in the sector $|\arg(\lambda)| \leqslant \pi - \delta$.

Proof. The function $L(a, \lambda)$ is defined as an integral over $0 \leqslant t < +\infty$ in theorem 6. We will rotate the contour of integration using Cauchy's integral formula. In doing so, we need to justify that the integrand in the definition of $L(a, \lambda)$ is analytic in some domain in the complex plane.

Let $0 < \delta < \frac{\pi}{m+2}$ be a fixed number. Suppose that $0 \leqslant \arg(\lambda) \leqslant \pi - \delta$. Then, if $0 \leqslant \arg(t) \leqslant \frac{1}{m}\arg(\lambda)$, there exists $M_0 > 0$ such that

$$-\pi < -\frac{\delta}{2} \leqslant \arg(t^m + P_{m-1}(t)) \leqslant \arg(\lambda) + \frac{\delta}{2} \leqslant \pi - \frac{\delta}{2},$$

provided that $|t| \geqslant M_0$. Since $t^m + P_{m-1}(t)$ lies in a large disc centred at the origin for $|t| \leqslant M_0$, we see that for all λ with $|\lambda|$ large, we have that $-\frac{\delta}{2} < \arg(t^m + P_{m-1}(t) + \lambda) < \pi - \frac{\delta}{2}$ and $|t^m + P_{m-1}(t) + \lambda| > 0$ for all t in the sector $0 \leqslant \arg(t) \leqslant \frac{1}{m}\arg(\lambda)$, and hence $\sqrt{t^m + P_{m-1}(t) + \lambda}$ is analytic in the sector $0 \leqslant \arg(t) \leqslant \frac{1}{m}\arg(\lambda)$ if λ lies outside a large disc and in the sector $0 \leqslant \arg(\lambda) \leqslant \pi - \delta$.

Let

$$Q(t, a, \lambda) = \begin{cases} \sqrt{t^m + P_{m-1}(t) + \lambda} - t^{\frac{m}{2}} - \sum_{j=1}^{\frac{m+1}{2}} b_j(a) t^{\frac{m}{2} - j}, & \text{if } m \text{ is odd,} \\ \sqrt{t^m + P_{m-1}(t) + \lambda} - t^{\frac{m}{2}} - \sum_{j=1}^{\frac{m}{2}} b_j(a) t^{\frac{m}{2} - j} - \frac{b_{\frac{m}{2}+1}(a)}{t+1}, & \text{if } m \text{ is even.} \end{cases}$$

Then, since $|Q(t, a, \lambda)| = O(|t|^{-\frac{m}{2}})$ as t tends to infinity in the sector $0 \leqslant \arg(t) \leqslant \frac{1}{m}\arg(\lambda)$, we have by Cauchy's integral formula, upon substituting $t = \lambda^{\frac{1}{m}}\tau$ for all λ with $|\lambda|$ large enough,

$$L(a, \lambda) = \int_0^{+\infty} Q(t, a, \lambda)\,\mathrm{d}t = \lambda^{\frac{1}{m}} \int_0^{+\infty} Q\big(\lambda^{\frac{1}{m}}\tau, a, \lambda\big)\,\mathrm{d}\tau, \tag{13}$$

where

$$Q\big(\lambda^{\frac{1}{m}}\tau, a, \lambda\big) = \begin{cases} \lambda^{\frac{1}{2}} \left(\sqrt{\tau^m + 1 + \dfrac{P_{m-1}\big(\lambda^{\frac{1}{m}}\tau\big)}{\lambda}} - \tau^{\frac{m}{2}} - \sum_{j=1}^{\frac{m+1}{2}} b_j(a) \dfrac{\tau^{\frac{m}{2}-j}}{\lambda^{\frac{j}{m}}} \right), \\ \text{if } m \text{ is odd,} \\ \lambda^{\frac{1}{2}} \left(\sqrt{\tau^m + 1 + \dfrac{P_{m-1}\big(\lambda^{\frac{1}{m}}\tau\big)}{\lambda}} - \tau^{\frac{m}{2}} - \sum_{j=1}^{\frac{m}{2}} b_j(a) \dfrac{\tau^{\frac{m}{2}-j}}{\lambda^{\frac{j}{m}}} - \dfrac{\lambda^{-\frac{1}{2}} b_{\frac{m}{2}+1}(a)}{\lambda^{\frac{1}{m}}\tau + 1} \right), \\ \text{if } m \text{ is even.} \end{cases}$$

Similarly, (13) holds for $-\pi + \delta \leqslant \arg(\lambda) \leqslant 0$.

Next, we examine the following square root in $Q\big(\lambda^{\frac{1}{m}}\tau, a, \lambda\big)$:

$$\begin{aligned} \sqrt{\tau^m + 1 + \frac{P_{m-1}\big(\lambda^{\frac{1}{m}}\tau\big)}{\lambda}} &= \sqrt{\tau^m + 1}\sqrt{1 + \frac{P_{m-1}\big(\lambda^{\frac{1}{m}}\tau\big)}{\lambda(\tau^m + 1)}} \\ &= \sqrt{\tau^m + 1}\left(1 + \sum_{k=1}^{\infty} \binom{\frac{1}{2}}{k} \left(\frac{P_{m-1}\big(\lambda^{\frac{1}{m}}\tau\big)}{\lambda(\tau^m + 1)}\right)^k\right) \\ &\overset{\text{let}}{=} \sqrt{\tau^m + 1} + \sum_{j=1}^{\infty} \frac{g_j(\tau)}{\lambda^{\frac{j}{m}}}, \end{aligned}$$

where $g_j(\tau)$ are functions such that $g_j(\tau)$ are all integrable on $[0, R]$ for any $R > 0$. Moreover, by the definition of b_j in (8), we see that for $1 \leqslant j \leqslant m - 1$,

$$g_j(\tau) = \sum_{k=1}^{j} \frac{b_{j,k}(a)\tau^{mk-j}}{(\tau^m + 1)^{k-\frac{1}{2}}} \text{ for some constants } b_{j,k}(a) \text{ such that } \sum_{k=1}^{j} b_{j,k}(a) = b_j(a).$$

Thus,

$$\begin{aligned} g_j(\tau) - b_j(a)\tau^{\frac{m}{2}-j} &= \sum_{k=1}^{j} b_{j,k}(a)\left(\frac{\tau^{mk-j}}{(\tau^m+1)^{k-\frac{1}{2}}} - \tau^{\frac{m}{2}-j}\right) \\ &\underset{\tau\to\infty}{=} \sum_{k=1}^{j} b_{j,k}(a)\tau^{\frac{m}{2}-j} O\left(\frac{1}{\tau^m}\right) \\ &\underset{\tau\to\infty}{=} O\left(\frac{1}{\tau^{\frac{m}{2}+j}}\right), \qquad \text{for all } 1 \leqslant j \leqslant \frac{m+1}{2}. \end{aligned}$$

So $\int_0^\infty \left|g_j(\tau) - b_j(a)\tau^{\frac{m}{2}-j}\right| \mathrm{d}\tau < +\infty$ for all $1 \leqslant j \leqslant \frac{m+1}{2}$. Next, when m is even and $j = \frac{m}{2}+1$, we write

$$\begin{aligned} &\int_0^\infty \left(g_{\frac{m}{2}+1}(\tau) - \frac{b_{\frac{m}{2}+1}(a)}{\tau + \lambda^{-\frac{1}{m}}}\right)\mathrm{d}\tau \\ &\quad = \int_0^\infty \left(g_{\frac{m}{2}+1}(\tau) - \frac{b_{\frac{m}{2}+1}(a)}{\tau+1}\right)\mathrm{d}\tau + b_{\frac{m}{2}+1}(a)\int_0^\infty \left(\frac{1}{\tau+1} - \frac{1}{\tau+\lambda^{-\frac{1}{m}}}\right)\mathrm{d}\tau \\ &\quad \overset{\text{let}}{=} K_{m,\frac{m}{2}+1}(a) - \frac{b_{\frac{m}{2}+1}(a)}{m}\ln(\lambda), \end{aligned}$$

where we take $\mathrm{Im}(\ln(\lambda)) = \arg(\lambda) \in (-\pi, \pi)$.

Thus, we have that

$$L(a,\lambda) = \begin{cases} \sum_{j=0}^{\frac{m+1}{2}} K_{m,j}(a)\lambda^{\frac{1}{2}+\frac{1-j}{m}} + O\left(|\lambda|^{-\frac{1}{2m}}\right), & \text{if } m \text{ is odd}, \\ \sum_{j=0}^{\frac{m}{2}+1} K_{m,j}(a)\lambda^{\frac{1}{2}+\frac{1-j}{m}} - \frac{b_{\frac{m}{2}+1}(a)}{m}\ln(\lambda) + O\left(|\lambda|^{-\frac{1}{m}}\right), & \text{if } m \text{ is even}, \end{cases}$$

as $\lambda \to \infty$ in the sector $|\arg(\lambda)| \leqslant \pi - \delta$, where

$$\begin{aligned} K_{m,0}(a) = K_m &= \int_0^\infty \left(\sqrt{1+t^m} - \sqrt{t^m}\right)\mathrm{d}t > 0, && \text{for all } m \geqslant 3, \\ K_{m,j}(a) &= \int_0^\infty \left(g_j(t) - b_j(a)t^{\frac{m}{2}-j}\right)\mathrm{d}t, && \text{for all } 1 \leqslant j \leqslant \frac{m+1}{2}, \\ K_{m,\frac{m}{2}+1}(a) &= \int_0^\infty \left(g_{\frac{m}{2}+1}(t) - \frac{b_{\frac{m}{2}+1}(a)}{t+1}\right)\mathrm{d}t, && \text{when } m \text{ is even}. \end{aligned} \tag{14}$$

This completes the proof. □

3. Eigenvalues are zeros of an entire function

In this section, we will prove that the eigenvalues are zeros of an entire function, due to Sibuya [19].

First, we let

$$G^k(a) := (\omega^{-k}a_1, \omega^{-2k}a_2, \ldots, \omega^{-(m-1)k}a_{m-1}), \qquad \text{for} \quad k \in \mathbb{Z}.$$

Then, recall that the function $f(z,a,\lambda)$ in theorem 6 solves (6) and decays to zero exponentially as $z \to \infty$ in S_0, and blows up in $S_{-1} \cup S_1$. Next, one can check that the function

$$f_k(z,a,\lambda) := f(\omega^{-k}z, G^k(a), \omega^{-mk}\lambda),$$

which is obtained by scaling $f(z, G^k(a), \omega^{-mk}\lambda)$ in the z-variable, also solves (6). It is clear that $f_0(z,a,\lambda) = f(z,a,\lambda)$. Also, $f_k(z,a,\lambda)$ decays in S_k and blows up in $S_{k-1} \cup S_{k+1}$ since $f(z, G^k(a), \omega^{-mk}\lambda)$ decays in S_0. Since no nonconstant solution decays in two consecutive

Stokes sectors (see lemma 5 (ii)), f_k and f_{k+1} are linearly independent and hence any solution of (6) can be expressed as a linear combination of these two. Especially, there exist some coefficients $C(a,\lambda)$ and $\widetilde{C}(a,\lambda)$ such that

$$f_{-1}(z,a,\lambda) = C(a,\lambda) f_0(z,a,\lambda) + \widetilde{C}(a,\lambda) f_1(z,a,\lambda). \tag{15}$$

We then see that

$$C(a,\lambda) = \frac{W_{-1,1}(a,\lambda)}{W_{0,1}(a,\lambda)} \qquad \text{and} \qquad \widetilde{C}(a,\lambda) = -\frac{W_{-1,0}(a,\lambda)}{W_{0,1}(a,\lambda)}, \tag{16}$$

where $W_{j,k} = f_j f_k' - f_j' f_k$ is the Wronskian of f_j and f_k. Since both f_j and f_k are solutions of the same linear equation (6), we know that the Wronskians are constant functions of z. Also, since f_k and f_{k+1} are linearly independent, $W_{k,k+1} \neq 0$ for all $k \in \mathbb{Z}$. Moreover, we have the following lemma that is useful later on.

Lemma 9. *Suppose $k, j \in \mathbb{Z}$. Then,*

$$W_{k+1,j+1}(a,\lambda) = \omega^{-1} W_{k,j}(G(a), \omega^2\lambda) \tag{17}$$

and $W_{0,1}(a,\lambda) = 2\omega^{\mu(a)}$, where

$$\mu(a) = \begin{cases} \frac{m}{4}, & \text{if } m \text{ is odd,} \\ \frac{m}{4} - b_{\frac{m}{2}+1}(a), & \text{if } m \text{ is even.} \end{cases}$$

Moreover,

$$\widetilde{C}(a,\lambda) = -\frac{W_{-1,0}(a,\lambda)}{W_{0,1}(a,\lambda)} = -\omega \frac{W_{0,1}(G^{-1}(a), \omega^{-2}\lambda)}{W_{0,1}(a,\lambda)} = -\omega^{1+2\nu(a)},$$

where

$$\nu(a) = \begin{cases} 0, & \text{if } m \text{ is odd,} \\ b_{\frac{m}{2}+1}(a), & \text{if } m \text{ is even.} \end{cases} \tag{18}$$

Proof. See Sibuya [19, pp 116–8] for a proof. Here, we mention that by (8), we have $b_{\frac{m}{2}+1}(G^{-1}(a)) = -b_{\frac{m}{2}+1}(a)$ and hence $\nu(G^{-1}(a)) = -\nu(a)$. □

Now we can identify the eigenvalues of H as the zeros of the entire function $\lambda \mapsto C(a,\lambda)$.

Theorem 10. *For each fixed $a \in \mathbb{C}^{m-1}$, the function $\lambda \mapsto C(a,\lambda)$ is entire. Moreover, λ is an eigenvalue of H if and only if $C(a,\lambda) = 0$.*

Proof. Since $W_{0,1}(a,\lambda) \neq 0$ and since $W_{-1,1}(a,\lambda)$ is a Wronskian of two entire functions, it is clear from (16) that $C(a,\lambda)$ is an entire function of λ for each fixed $a \in \mathbb{C}^{m-1}$.

Next, suppose that λ is an eigenvalue of H with a corresponding eigenfunction u. Then the scaled eigenfunction $v(z,\lambda) = u(-iz,\lambda)$ solves (6) and decays in $S_{-1} \cup S_1$. Hence, v is a (nonzero) constant multiple of f_1 since both decay in S_1. Similarly, v is also a constant multiple of f_{-1}. Thus, f_{-1} is a constant multiple of f_1, implying $C(a,\lambda) = 0$.

Conversely, if $C(a,\lambda) = 0$, then f_{-1} is a constant multiple of f_1, and hence f_1 also decays in S_{-1}. Thus, f_1 decays in $S_{-1} \cup S_1$ and is a scaled eigenfunction with the eigenvalue λ. □

Moreover, the following is an easy consequence of (15): for each $k \in \mathbb{Z}$, we have

$$W_{-1,k}(a,\lambda) = C(a,\lambda) W_{0,k}(a,\lambda) + \widetilde{C}(a) W_{1,k}(a,\lambda), \tag{19}$$

where we use $\widetilde{C}(a)$ for $\widetilde{C}(a,\lambda)$ since it is independent of λ.

4. Asymptotic expansions of $C(a, \lambda)$

In this section, we provide asymptotic expansions of the entire function $C(a, \lambda)$ as $\lambda \to \infty$ along all possible rays to infinity in the complex plane.

First, we provide an asymptotic expansion of the Wronskian of f_0 and f_j in preparation for providing an asymptotic expansion of $C(a, \lambda)$.

Lemma 11. *Suppose that* $1 \leqslant j \leqslant \frac{m}{2} + 1$. *Then, for each* $a \in \mathbb{C}^{m-1}$ *and* $0 < \delta < \frac{\pi}{m+2}$,

$$W_{0,j}(a, \lambda) = \left[2\mathrm{i}\omega^{-\frac{j}{2}} + o(1)\right] \exp[L(G^j(a), \omega^{2j-m-2}\lambda) + L(a, \lambda)], \tag{20}$$

as $\lambda \to \infty$ *in the sector*

$$-\pi + \delta \leqslant \pi - \frac{4j\pi}{m+2} + \delta \leqslant \arg(\lambda) \leqslant \pi - \delta. \tag{21}$$

Proof. We fix $1 \leqslant j \leqslant \frac{m}{2} + 1$. Then,

$$\begin{aligned} W_{0,j}(a, \lambda) &= f_0(z, a, \lambda) f_j'(z, a, \lambda) - f_0'(z, a, \lambda) f_j(z, a, \lambda) \\ &= \omega^{-j} f(0, a, \lambda) f'(0, G^j(a), \omega^{2j-m-2}\lambda) - f'(0, a, \lambda) f(0, G^j(a), \omega^{2j-m-2}\lambda) \\ &\underset{\lambda\to\infty}{=} -\left[\omega^{-j}\omega^{\frac{2j-m-2}{4}} - \omega^{-\frac{2j-m-2}{4}} + o(1)\right] \exp[L(G^j(a), \omega^{2j-m-2}\lambda) + L(a, \lambda)] \\ &\underset{\lambda\to\infty}{=} \left[2\mathrm{i}\omega^{-\frac{j}{2}} + o(1)\right] \exp[L(G^j(a), \omega^{2j-m-2}\lambda) + L(a, \lambda)], \end{aligned}$$

where we used (9) and (10) with

$$|\arg(\lambda)| \leqslant \pi - \delta \quad \text{and} \quad |\arg(\omega^{2j-m-2}\lambda)| \leqslant \pi - \delta,$$

which is (21). Here we also used $j \leqslant \frac{m}{2} + 1$. □

Next, we provide an asymptotic expansion of $W_{-1,1}(a, \lambda)$ as $\lambda \to \infty$ along the rays near the negative real axis. Note from (16) that $W_{-1,1}(a, \lambda) = W_{0,1}(a, \lambda)C(a, \lambda)$. Also, $W_{0,1}(a, \lambda)$ is a nonzero constant function of λ. So from these one gets an asymptotic expansion of $C(a, \lambda)$.

Theorem 12. *For each fixed* $a \in \mathbb{C}^{m-1}$ *and* $0 < \delta < \frac{\pi}{m+2}$,

$$W_{-1,1}(a, \lambda) = [2\mathrm{i} + o(1)] \exp[L(G^{-1}(a), \omega^{-2}\lambda) + L(G(a), \omega^{-m}\lambda)], \tag{22}$$

as $\lambda \to \infty$ *along the rays in the sector*

$$\pi - \frac{4\pi}{m+2} + \delta \leqslant \arg(\lambda) \leqslant \pi + \frac{4\pi}{m+2} - \delta. \tag{23}$$

Moreover, there exists a constant $M_1 > 0$ *such that* $W_{-1,1}(a, \lambda) \neq 0$ *for all* λ *in the sector (23) if* $|\lambda| \geqslant M_1$.

Proof. This is an easy consequence of lemma 11 and equation (17).

The last assertion of the theorem is a consequence of the asymptotic expansion (22). □

The asymptotic expansion of $C(a, \lambda)$ in a sector near the positive real axis is obtained in the following theorem.

Theorem 13. *Suppose that* $m \geqslant 4$. *Then, for each fixed* $a \in \mathbb{C}^{m-1}$ *and* $0 < \delta < \frac{\pi}{m+2}$,

$$\begin{aligned} C(a, \lambda) = &\left[\omega^{\frac{1}{2}} + o(1)\right] \exp[L(G^{-1}(a), \omega^{-2}\lambda) - L(a, \lambda)] \\ &+ \left[\omega^{\frac{1}{2}+2\nu(a)} + o(1)\right] \exp[L(G(a), \omega^{2}\lambda) - L(a, \lambda)], \end{aligned}$$

as $\lambda \to \infty$ *in the sector*

$$\pi - \frac{4\lfloor\frac{m}{2}\rfloor\pi}{m+2} + \delta \leqslant \arg(\lambda) \leqslant \pi - \frac{4\pi}{m+2} - \delta. \tag{24}$$

Proof. Suppose $2 \leqslant k \leqslant \frac{m}{2}$. Then, from (17), (19) and lemma 11,

$$\begin{aligned}
C(a,\lambda) &= \frac{W_{-1,k}(a,\lambda)}{W_{0,k}(a,\lambda)} - \widetilde{C}(a)\frac{W_{1,k}(a,\lambda)}{W_{0,k}(a,\lambda)} \\
&= \frac{\omega W_{0,k+1}(G^{-1}(a),\omega^{-2}\lambda)}{W_{0,k}(a,\lambda)} - \widetilde{C}(a)\frac{\omega^{-1}W_{0,k-1}(G(a),\omega^2\lambda)}{W_{0,k}(a,\lambda)} \\
&= \left[\omega^{\frac{1}{2}} + o(1)\right]\frac{\exp[L(G^k(a),\omega^{2k-m-2}\lambda) + L(G^{-1}(a),\omega^{-2}\lambda)]}{\exp[L(G^k(a),\omega^{2k-m-2}\lambda) + L(a,\lambda)]} \\
&\quad - \left[\omega^{-\frac{1}{2}} + o(1)\right]\widetilde{C}(a)\frac{\exp[L(G^k(a),\omega^{2k-m-2}\lambda) + L(G(a),\omega^2\lambda)]}{\exp[L(G^k(a),\omega^{2k-m-2}\lambda) + L(a,\lambda)]} \\
&= \left[\omega^{\frac{1}{2}} + o(1)\right]\exp[L(G^{-1}(a),\omega^{-2}\lambda) - L(a,\lambda)] \\
&\quad + \left[\omega^{-\frac{1}{2}} + o(1)\right]\omega^{1+2\nu(a)}\exp[L(G(a),\omega^2\lambda) - L(a,\lambda)],
\end{aligned}$$

as $\lambda \to \infty$ such that

$$\begin{aligned}
-\pi < \pi - \frac{4(k+1)\pi}{m+2} + \delta &\leqslant \arg(\omega^{-2}\lambda) \leqslant \pi - \delta, \\
\pi - \frac{4k\pi}{m+2} + \delta &\leqslant \arg(\lambda) \leqslant \pi - \delta, \\
\pi - \frac{4(k-1)\pi}{m+2} + \delta &\leqslant \arg(\omega^2\lambda) \leqslant \pi - \delta,
\end{aligned}$$

that is,

$$\pi - \frac{4k\pi}{m+2} + \delta \leqslant \arg(\lambda) \leqslant \pi - \frac{4\pi}{m+2} - \delta,$$

provided that $2 \leqslant k \leqslant \frac{m}{2}$. So in order to complete the proof, we choose $k = \lfloor\frac{m}{2}\rfloor$. □

The sectors (23) and (24) do not cover the entire complex plane. The next theorem covers a sector in the upper-half plane, connecting the sectors (23) and (24) in the upper-half plane.

Theorem 14. *Suppose that* $a \in \mathbb{C}^{m-1}$ *and* $0 < \delta < \frac{\pi}{m+2}$. *If* $m \geqslant 4$, *then*

$$\begin{aligned}
C(a,\lambda) &= \left[\omega^{\frac{1}{2}} + o(1)\right]\exp[L(G^{-1}(a),\omega^{-2}\lambda) - L(a,\lambda)] \\
&\quad - [\mathrm{i}\omega^{1+\mu(a)+4\nu(a)} + o(1)]\exp[-L(G^2(a),\omega^{2-m}\lambda) - L(a,\lambda)],
\end{aligned} \tag{25}$$

as $\lambda \to \infty$ *in the sector*

$$\pi - \frac{8\pi}{m+2} + \delta \leqslant \arg(\lambda) \leqslant \pi - \delta. \tag{26}$$

If $m = 3$, *then*

$$\begin{aligned}
C(a,\lambda) &= [-\omega^{-2} + o(1)]\exp[L(G^4(a),\omega^{-2}\lambda) - L(a,\lambda)] \\
&\quad - \left[\mathrm{i}\omega^{\frac{7}{4}} + o(1)\right]\exp[-L(G^2(a),\omega^{-1}\lambda) - L(a,\lambda)],
\end{aligned}$$

as $\lambda \to \infty$ *in the sector*

$$-\frac{\pi}{5} + \delta \leqslant \arg(\lambda) \leqslant \pi - \delta. \tag{27}$$

Moreover, if $m \geqslant 6$, then there exists a constant $M_2 > 0$ such that $C(a, \lambda) \neq 0$ for all λ in the sector (26) if $|\lambda| \geqslant M_2$.

Proof. Suppose that $m \geqslant 4$. Then, from lemmas 9 and 11, and (19) with $k = 2$,

$$\begin{aligned}
C(a,\lambda) &= \frac{W_{-1,2}(a,\lambda)}{W_{0,2}(a,\lambda)} - \widetilde{C}(a)\frac{W_{1,2}(a,\lambda)}{W_{0,2}(a,\lambda)}\\
&= \frac{\omega W_{0,3}(G^{-1}(a),\omega^{-2}\lambda)}{W_{0,2}(a,\lambda)} - \widetilde{C}(a)\frac{\omega^{-1}W_{0,1}(G(a),\omega^{2}\lambda)}{W_{0,2}(a,\lambda)}\\
&= \frac{\omega\left[2\mathrm{i}\omega^{-\frac{3}{2}} + o(1)\right]\exp[L(G^2(a),\omega^{2-m}\lambda) + L(G^{-1}(a),\omega^{-2}\lambda)]}{\left[2\mathrm{i}\omega^{-\frac{2}{2}} + o(1)\right]\exp[L(G^2(a),\omega^{2-m}\lambda) + L(a,\lambda)]}\\
&\quad - \widetilde{C}(a)\frac{\omega^{-1}W_{0,1}(G(a),\omega^{2}\lambda)}{[2\mathrm{i}\omega^{-1} + o(1)]\exp[L(G^2(a),\omega^{2-m}\lambda) + L(a,\lambda)]}\\
&= \left[\omega^{\frac{1}{2}} + o(1)\right]\exp[L(G^{-1}(a),\omega^{-2}\lambda) - L(a,\lambda)]\\
&\quad - \frac{-\omega^{1+2\nu(a)}2\omega^{\mu(G(a))}}{[2\mathrm{i} + o(1)]\exp[L(G^2(a),\omega^{2-m}\lambda) + L(a,\lambda)]},
\end{aligned}$$

as $\lambda \to \infty$ such that

$$-\pi + \delta \leqslant \pi - \frac{12\pi}{m+2} + \delta \leqslant \arg(\omega^{-2}\lambda) \leqslant \pi - \delta \quad \text{and} \quad \pi - \frac{8\pi}{m+2} + \delta \leqslant \arg(\lambda) \leqslant \pi - \delta,$$

that is,

$$\pi - \frac{8\pi}{m+2} + \delta \leqslant \arg(\lambda) \leqslant \pi - \delta.$$

Next, we use $2\nu(a) + \mu(G(a)) = \mu(a) + 4\nu(a)$ to get (25).

Suppose $m = 3$. Then, $\omega^5 = 1$. Also, $W_{-3,0}(a,\lambda) = W_{2,0}(a,\lambda)$ since $f_{-3}(z,a,\lambda) = f_2(z,a,\lambda)$. Thus, we have that

$$\begin{aligned}
C(a,\lambda) &= \frac{W_{-1,2}(a,\lambda)}{W_{0,2}(a,\lambda)} - \widetilde{C}(a)\frac{W_{1,2}(a,\lambda)}{W_{0,2}(a,\lambda)}\\
&= \frac{\omega^{-2}W_{-3,0}(G^2(a),\omega^{4}\lambda)}{W_{0,2}(a,\lambda)} - \widetilde{C}(a)\frac{\omega^{-1}W_{0,1}(G(a),\omega^{2}\lambda)}{W_{0,2}(a,\lambda)}\\
&= -\frac{\omega^{-2}W_{0,2}(G^2(a),\omega^{4}\lambda)}{W_{0,2}(a,\lambda)} - \widetilde{C}(a)\frac{\omega^{-1}W_{0,1}(G(a),\omega^{2}\lambda)}{W_{0,2}(a,\lambda)}\\
&= -\frac{\omega^{-2}W_{0,2}(G^2(a),\omega^{-1}\lambda)}{W_{0,2}(a,\lambda)} - \widetilde{C}(a)\frac{\omega^{-1}W_{0,1}(G(a),\omega^{2}\lambda)}{W_{0,2}(a,\lambda)}\\
&= -\frac{\omega^{-2}\left[2\mathrm{i}\omega^{-\frac{2}{2}} + o(1)\right]\exp[L(G^4(a),\omega^{-2}\lambda) + L(G^2(a),\omega^{-1}\lambda)]}{\left[2\mathrm{i}\omega^{-\frac{2}{2}} + o(1)\right]\exp[L(G^2(a),\omega^{-1}\lambda) + L(a,\lambda)]}\\
&\quad - \widetilde{C}(a)\frac{\omega^{-1}W_{0,1}(G(a),\omega^{2}\lambda)}{[2\mathrm{i}\omega^{-1} + o(1)]\exp[L(G^2(a),\omega^{-1}\lambda) + L(a,\lambda)]}\\
&= [-\omega^{-2} + o(1)]\exp[L(G^4(a),\omega^{-2}\lambda) - L(a,\lambda)]\\
&\quad - \frac{-\omega^{1+2\nu(a)}2\omega^{\mu(G(a))}}{[2\mathrm{i} + o(1)]\exp[L(G^2(a),\omega^{-1}\lambda) + L(a,\lambda)]},
\end{aligned}$$

as $\lambda \to \infty$ such that

$$-\pi + \delta \leqslant \pi - \frac{8\pi}{5} + \delta \leqslant \arg(\omega^{-1}\lambda) \leqslant \pi - \delta \qquad \text{and} \qquad \pi - \frac{8\pi}{5} + \delta \leqslant \arg(\lambda) \leqslant \pi - \delta,$$

that is,

$$\pi - \frac{6\pi}{5} + \delta \leqslant \arg(\lambda) \leqslant \pi - \delta.$$

In order to show the last assertion, we suppose that $C(a,\lambda) = 0$ for some λ in (26) with large $|\lambda|$. Then, from the asymptotic expansion (25), we have

$$\exp[L(G^{-1}(a), \omega^{-2}\lambda) + L(G^2(a), \omega^{2-m}\lambda)] = \mathrm{i}\omega^{\frac{1}{2}+\mu(a)+4\nu(a)} + o(1) \tag{28}$$

By corollary 7,

$$\begin{aligned}
\mathrm{Re}(L(G^{-1}(a), \omega^{-2}\lambda)) &= K_m \cos\left(\frac{m+2}{2m}\arg(\omega^{-2}\lambda)\right)|\lambda|^{\frac{1}{2}+\frac{1}{m}}(1+o(1))\\
&= K_m \cos\left(-\frac{2\pi}{m} + \frac{m+2}{2m}\arg(\lambda)\right)|\lambda|^{\frac{1}{2}+\frac{1}{m}}(1+o(1)),\\
\mathrm{Re}(L(G^2(a), \omega^{2-m}\lambda)) &= K_m \cos\left(\frac{m+2}{2m}\arg(\omega^{2-m}\lambda)\right)|\lambda|^{\frac{1}{2}+\frac{1}{m}}(1+o(1))\\
&= -K_m \cos\left(\frac{2\pi}{m} + \frac{m+2}{2m}\arg(\lambda)\right)|\lambda|^{\frac{1}{2}+\frac{1}{m}}(1+o(1)).
\end{aligned}$$

Note that if $m \geqslant 6$, then $0 < \delta \leqslant \arg(\lambda) \leqslant \pi - \delta$ in (26). Since

$$\begin{aligned}
&\cos\left(-\frac{2\pi}{m} + \frac{m+2}{2m}\arg(\lambda)\right) - \cos\left(\frac{2\pi}{m} + \frac{m+2}{2m}\arg(\lambda)\right)\\
&\qquad = 2\sin\left(\frac{2\pi}{m}\right)\sin\left(\frac{m+2}{2m}\arg(\lambda)\right) > 0,
\end{aligned}$$

we see that

$$\mathrm{Re}(L(G^{-1}(a), \omega^{-2}\lambda) + L(G^2(a), \omega^{2-m}\lambda)) \to +\infty,$$

as $\lambda \to \infty$ in (26), and hence the left-hand side of (28) blows up. Thus, $C(a,\lambda)$ cannot have infinitely many zeros in (26). This completes the proof. □

The next theorem covers a sector in the lower-half plane, connecting sectors (23) and (24).

Theorem 15. *Suppose that* $a \in \mathbb{C}^{m-1}$ *and* $0 < \delta < \frac{\pi}{m+2}$. *If* $m \geqslant 4$, *then*

$$\begin{aligned}
C(a,\lambda) = {} & [-\mathrm{i}\omega^{1+\mu(a)} + o(1)]\exp[-L(a, \omega^{-m-2}\lambda) - L(G^{-2}(a), \omega^{-4}\lambda)]\\
& + \left[\omega^{\frac{1}{2}+2\nu(a)} + o(1)\right]\exp[L(G(a), \omega^{-m}\lambda) - L(a, \omega^{-m-2}\lambda)],
\end{aligned}$$

as $\lambda \to \infty$ *in the sector*

$$\pi + \delta \leqslant \arg(\lambda) \leqslant \pi + \frac{8\pi}{m+2} - \delta. \tag{29}$$

If $m = 3$, *then*

$$\begin{aligned}
C(a,\lambda) = {} & \left[-\mathrm{i}\omega^{\frac{7}{4}} + o(1)\right]\exp[-L(a, \omega^{-5}\lambda) - L(G^{-2}(a), \omega^{-4}\lambda)]\\
& + [\omega^3 + o(1)]\exp[L(G^{-1}(a), \omega^{-3}\lambda) - L(a, \omega^{-5}\lambda)],
\end{aligned}$$

as $\lambda \to \infty$ *in the sector*

$$\pi + \delta \leqslant \arg(\lambda) \leqslant 2\pi + \frac{\pi}{5} - \delta. \tag{30}$$

Moreover, if $m \geqslant 6$, *then there exists a constant* $M_3 > 0$ *such that* $C(a,\lambda) \neq 0$ *for all* λ *in the sector* (29) *if* $|\lambda| \geqslant M_3$.

Proof. Suppose that $m \geqslant 4$. Then, from lemmas 9 and 11, and (19) with $k = -2$,

$$\begin{aligned}
C(a,\lambda) &= \frac{W_{-1,-2}(a,\lambda)}{W_{0,-2}(a,\lambda)} - \widetilde{C}(a)\frac{W_{1,-2}(a,\lambda)}{W_{0,-2}(a,\lambda)} \\
&= \frac{W_{0,1}(G^{-2}(a),\omega^{-4}\lambda)}{W_{0,2}(G^{-2}(a),\omega^{-4}\lambda)} - \widetilde{C}(a)\frac{W_{0,3}(G^{-2}(a),\omega^{-4}\lambda)}{W_{0,2}(G^{-2}(a),\omega^{-4}\lambda)} \\
&= \frac{W_{0,1}(G^{-2}(a),\omega^{-4}\lambda)}{\left[2\mathrm{i}\omega^{-\frac{2}{2}}+o(1)\right]\exp[L(a,\omega^{-m-2}\lambda)+L(G^{-2}(a),\omega^{-4}\lambda)]} \\
&\quad - \widetilde{C}(a)\frac{\left[2\mathrm{i}\omega^{-\frac{3}{2}}+o(1)\right]\exp[L(G(a),\omega^{-m}\lambda)+L(G^{-2}(a),\omega^{-4}\lambda)]}{\left[2\mathrm{i}\omega^{-\frac{2}{2}}+o(1)\right]\exp[L(a,\omega^{-m-2}\lambda)+L(G^{-2}(a),\omega^{-4}\lambda)]} \\
&= \frac{2\omega^{\mu(G^{-2}(a))}}{[2\mathrm{i}\omega^{-1}+o(1)]\exp[L(a,\omega^{-m-2}\lambda)+L(G^{-2}(a),\omega^{-4}\lambda)]} \\
&\quad + \left[\omega^{-\frac{1}{2}}+o(1)\right]\omega^{1+2\nu(a)}\frac{\exp[L(G(a),\omega^{-m}\lambda)+L(G^{-2}(a),\omega^{-4}\lambda)]}{\exp[L(a,\omega^{-m-2}\lambda)+L(G^{-2}(a),\omega^{-4}\lambda)]},
\end{aligned}$$

as $\lambda \to \infty$ such that

$$\pi - \frac{12\pi}{m+2} + \delta \leqslant \arg(\omega^{-4}\lambda) \leqslant \pi - \delta \quad \text{and} \quad \pi - \frac{8\pi}{m+2} + \delta \leqslant \arg(\omega^{-4}\lambda) \leqslant \pi - \delta, \tag{31}$$

that is,

$$\pi + \delta \leqslant \arg(\lambda) \leqslant \pi + \frac{8\pi}{m+2} - \delta, \tag{32}$$

which is (29).

Suppose that $m = 3$. Then,

$$\begin{aligned}
C(a,\lambda) &= \frac{W_{-1,-2}(a,\lambda)}{W_{0,-2}(a,\lambda)} - \widetilde{C}(a)\frac{W_{1,-2}(a,\lambda)}{W_{0,-2}(a,\lambda)} \\
&= \frac{W_{0,1}(G^{-2}(a),\omega^{-4}\lambda)}{W_{0,2}(G^{-2}(a),\omega^{-4}\lambda)} + \widetilde{C}(a)\frac{\omega^{-1}W_{0,-3}(G(a),\omega^{2}\lambda)}{\omega^{2}W_{0,2}(G^{-2}(a),\omega^{-4}\lambda)} \\
&= \frac{W_{0,1}(G^{-2}(a),\omega^{-4}\lambda)}{W_{0,2}(G^{-2}(a),\omega^{-4}\lambda)} + \omega^{2}\widetilde{C}(a)\frac{W_{0,2}(G^{-4}(a),\omega^{-3}\lambda)}{W_{0,2}(G^{-2}(a),\omega^{-4}\lambda)} \\
&= \frac{W_{0,1}(G^{-2}(a),\omega^{-4}\lambda)}{\left[2\mathrm{i}\omega^{-\frac{2}{2}}+o(1)\right]\exp[L(a,\omega^{-5}\lambda)+L(G^{-2}(a),\omega^{-4}\lambda)]} \\
&\quad - \omega^{2}\widetilde{C}(a)\frac{\left[2\mathrm{i}\omega^{-\frac{2}{2}}+o(1)\right]\exp[L(G^{-2}(a),\omega^{-4}\lambda)+L(G^{-4}(a),\omega^{-3}\lambda)]}{\left[2\mathrm{i}\omega^{-\frac{2}{2}}+o(1)\right]\exp[L(a,\omega^{-5}\lambda)+L(G^{-2}(a),\omega^{-4}\lambda)]} \\
&= \frac{2\omega^{\mu(G^{-2}(a))}}{[2\mathrm{i}\omega^{-1}+o(1)]\exp[L(a,\omega^{-5}\lambda)+L(G^{-2}(a),\omega^{-4}\lambda)]} \\
&\quad + [\omega^{2}+o(1)]\omega^{1+2\nu(a)}\frac{\exp[L(G^{-2}(a),\omega^{-4}\lambda)+L(G^{-4}(a),\omega^{-3}\lambda)]}{\exp[L(a,\omega^{-5}\lambda)+L(G^{-2}(a),\omega^{-4}\lambda)]},
\end{aligned}$$

as $\lambda \to \infty$ such that

$$\pi - \frac{8\pi}{5} + \delta \leqslant \arg(\omega^{-3}\lambda) \leqslant \pi - \delta \quad \text{and} \quad \pi - \frac{8\pi}{5} + \delta \leqslant \arg(\omega^{-4}\lambda) \leqslant \pi - \delta,$$

that is,

$$\pi + \delta \leqslant \arg(\lambda) \leqslant \pi + \frac{6\pi}{5} - \delta.$$

Finally, the proof of the last assertion of this theorem follows as in the proof of theorem 14. □

From the asymptotic expansions in the previous four theorems, one obtains the order of the entire function $\lambda \mapsto C(a,\lambda)$. The order of an entire function g is defined by

$$\limsup_{r\to\infty} \frac{\ln \ln M(r,g)}{\ln r},$$

where $M(r,g) = \max\{|g(r\,\mathrm{e}^{\mathrm{i}\theta})| : 0 \leqslant \theta \leqslant 2\pi\}$ for $r > 0$. If for some positive real numbers σ, c_1, c_2, we have $\exp[c_1 r^\sigma] \leqslant M(r,g) \leqslant \exp[c_2 r^\sigma]$ for all large r, then the order of g is σ.

Corollary 16. *The entire function* $\lambda \mapsto C(a,\lambda)$ *is of order* $\frac{1}{2} + \frac{1}{m}$.

Proof. The sectors in (21), (23), (26) and (29), cover the entire complex plane. So the nonconstant entire function $|C(a,\lambda)|$ is bounded above by $c_1 \exp\left[d_1|\lambda|^{\frac{1}{2}+\frac{1}{m}}\right]$ for some positive constants c_1, d_1. Also, along the ray $\arg(\lambda) = \pi$, one can see from (11) and (22) that $|C(a,\lambda)|$ is bounded below by $c_2 \exp\left[d_2|\lambda|^{\frac{1}{2}+\frac{1}{m}}\right]$ for some positive constants c_2, d_2. Hence, the order of $C(a,\cdot)$ is $\frac{1}{2} + \frac{1}{m}$. □

Remark. Since the eigenvalues are the zeros of the entire function $\lambda \mapsto C(a,\lambda)$ of order $\frac{1}{2} + \frac{1}{m} \in (0,1)$, there are infinitely many discrete eigenvalues as was already mentioned in theorem 1.

5. Asymptotic expansion of the eigenvalues: proof of theorem 2

In this section, we prove theorem 2 by using the asymptotic expansions of $C(a,\lambda)$ and $L(a,\lambda)$.

Proof of theorem 2. Recall that by theorem 10, λ is an eigenvalue of H if and only if $C(a,\lambda) = 0$.

For $m \geqslant 4$ and $a \in \mathbb{C}^{m-1}$ fixed, suppose that $C(a,\lambda) = 0$ for some λ with $|\lambda|$ large. Then, from the asymptotic expansion of $C(a,\lambda)$ in theorem 13, we have

$$[1 + o(1)]\exp[L(G(a),\omega^2\lambda) - L(G^{-1}(a),\omega^{-2}\lambda)] = -\omega^{-2\nu(a)},$$

and absorbing $[1 + o(1)]$ into the exponential function then yields

$$\exp[L(G(a),\omega^2\lambda) - L(G^{-1}(a),\omega^{-2}\lambda) + o(1)] = -\omega^{-2\nu(a)}.$$

Thus, from lemma 8 if m is odd, we infer

$$\begin{aligned}
\ln(-\omega^{-2\nu(a)}) &= L(G(a),\omega^2\lambda) - L(G^{-1}(a),\omega^{-2}\lambda) + o(1) \\
&= \sum_{j=0}^{\lfloor \frac{m}{2}+1 \rfloor} [K_{m,j}(G(a))(\omega^2\lambda)^{\frac{1}{2}+\frac{1-j}{m}} - K_{m,j}(G^{-1}(a))(\omega^{-2}\lambda)^{\frac{1}{2}+\frac{1-j}{m}}] + o(1) \\
&= 2\mathrm{i}K_{m,0}\sin\left(\frac{2\pi}{m}\right)\lambda^{\frac{1}{2}+\frac{1}{m}} + \sum_{j=1}^{\lfloor \frac{m}{2}+1 \rfloor} c_{m,j}(a)\lambda^{\frac{1}{2}+\frac{1-j}{m}} + o(1), \qquad (33)
\end{aligned}$$

where

$$c_{m,j}(a) = K_{m,j}(G(a))(\omega^2)^{\frac{1}{2}+\frac{1-j}{m}} - K_{m,j}(G^{-1}(a))(\omega^{-2})^{\frac{1}{2}+\frac{1-j}{m}}, \qquad 1 \leqslant j \leqslant \frac{m+1}{2}. \qquad (34)$$

Similarly, if m is even, then from lemma 8, we have (33) with $c_{m,j}(a)$ in (34) for $1 \leqslant j \leqslant \frac{m}{2}$, and

$$c_{m,\frac{m}{2}+1}(a) = K_{m,\frac{m}{2}+1}(G(a)) - K_{m,\frac{m}{2}+1}(G^{-1}(a)) + \frac{b_{\frac{m}{2}+1}(a)}{m}\frac{8\pi \mathrm{i}}{m+2},$$

where we used $b_{\frac{m}{2}+1}(G^{-1}(a)) = -b_{\frac{m}{2}+1}(a) = b_{\frac{m}{2}+1}(G(a))$.

Note that there exist constants $M_4 > 0$ and $\varepsilon > 0$ such that the function

$$\lambda \mapsto L(G(a), \omega^2\lambda) - L(G^{-1}(a), \omega^{-2}\lambda) + o(1) \tag{35}$$

is continuous in the region $|\lambda| \geqslant M_4$ and $|\arg(\lambda)| \leqslant \varepsilon$. From (33) we then see that the function (35) maps the region $|\lambda| \geqslant M_4$ and $|\arg(\lambda)| \leqslant \varepsilon$ onto a region that contains the entire positive imaginary axis near infinity.

Thus, from (33) we get that for every sufficiently large $n \in \mathbb{N}$ there exists λ_n such that

$$2\mathrm{i}K_{m,0}\sin\left(\frac{2\pi}{m}\right)\lambda_n^{\frac{1}{2}+\frac{1}{m}} + \sum_{j=1}^{\lfloor \frac{m}{2}+1\rfloor} c_{m,j}(a)\lambda_n^{\frac{1}{2}+\frac{1-j}{m}} + o(1) = \left(2n+1-\frac{4\nu(a)}{m+2}\right)\pi \mathrm{i}.$$

Thus,

$$\lambda_n^{\frac{1}{2}+\frac{1}{m}} + \sum_{j=1}^{\lfloor \frac{m}{2}+1\rfloor} \frac{c_{m,j}(a)}{2\mathrm{i}K_{m,0}\sin\left(\frac{2\pi}{m}\right)}\lambda_n^{\frac{1}{2}+\frac{1-j}{m}} + o(1) = \frac{\left(2n+1-\frac{4\nu(a)}{m+2}\right)\pi}{2K_{m,0}\sin\left(\frac{2\pi}{m}\right)}.$$

Let

$$d_{m,j}(a) = \begin{cases} \dfrac{c_{m,j}(a)}{2\mathrm{i}K_{m,0}\sin\left(\frac{2\pi}{m}\right)}, & \text{if } 1 \leqslant j \leqslant \dfrac{m+1}{2} \\[2ex] \dfrac{c_{m,j}(a) + \frac{4\nu(a)}{m+2}\pi \mathrm{i}}{2\mathrm{i}K_{m,0}\sin\left(\frac{2\pi}{m}\right)}, & \text{if } m \text{ is even and } j = \dfrac{m}{2}+1. \end{cases} \tag{36}$$

Then,

$$\lambda_n^{\frac{1}{2}+\frac{1}{m}} + \sum_{j=1}^{\lfloor \frac{m}{2}+1\rfloor} d_{m,j}(a)\lambda_n^{\frac{1}{2}+\frac{1-j}{m}} + o(1) = \frac{(2n+1)\pi}{2K_{m,0}\sin\left(\frac{2\pi}{m}\right)}. \tag{37}$$

Introduce the decomposition $\lambda_n = \lambda_{0,n} + \lambda_{1,n}$, where

$$\lambda_{0,n} = \left(\frac{(2n+1)\pi}{2K_{m,0}\sin\left(\frac{2\pi}{m}\right)}\right)^{\frac{2m}{m+2}} \qquad \text{and} \qquad \frac{\lambda_{1,n}}{\lambda_{0,n}} = o(1).$$

Then, from (37), we have

$$\begin{aligned} \lambda_{0,n}^{\frac{1}{2}+\frac{1}{m}} &= \lambda_{0,n}^{\frac{1}{2}+\frac{1}{m}}\left(1+\frac{\lambda_{1,n}}{\lambda_{0,n}}\right)^{\frac{1}{2}+\frac{1}{m}} + \sum_{j=1}^{\lfloor \frac{m}{2}+1\rfloor} d_{m,j}(a)\lambda_{0,n}^{\frac{1}{2}+\frac{1-j}{m}}\left(1+\frac{\lambda_{1,n}}{\lambda_{0,n}}\right)^{\frac{1}{2}+\frac{1-j}{m}} + o(1) \\ &= \lambda_{0,n}^{\frac{1}{2}+\frac{1}{m}}\left(1+\sum_{k=1}^{\infty}\binom{\frac{1}{2}+\frac{1}{m}}{k}\left(\frac{\lambda_{1,n}}{\lambda_{0,n}}\right)^k\right) \\ &\quad + \sum_{j=1}^{\lfloor \frac{m}{2}+1\rfloor} d_{m,j}(a)\lambda_{0,n}^{\frac{1}{2}+\frac{1-j}{m}}\left(1+\sum_{k=1}^{\infty}\binom{\frac{1}{2}+\frac{1-j}{m}}{k}\left(\frac{\lambda_{1,n}}{\lambda_{0,n}}\right)^k\right) + o(1). \end{aligned}$$

Thus,

$$0 = \left(\frac{1}{2} + \frac{1}{m}\right)\frac{\lambda_{1,n}}{\lambda_{0,n}} + \sum_{k=2}^{\infty}\binom{\frac{1}{2}+\frac{1}{m}}{k}\left(\frac{\lambda_{1,n}}{\lambda_{0,n}}\right)^k + \sum_{j=1}^{\lfloor\frac{m}{2}+1\rfloor} d_{m,j}(a)\lambda_{0,n}^{-\frac{j}{m}}\left(1+\sum_{k=1}^{\infty}\binom{\frac{1}{2}+\frac{1-j}{m}}{k}\left(\frac{\lambda_{1,n}}{\lambda_{0,n}}\right)^k\right) + o\big(\lambda_{0,n}^{-\frac{1}{2}-\frac{1}{m}}\big),$$

and hence

$$\frac{\lambda_{1,n}}{\lambda_{0,n}} + \frac{2m}{m+2}\sum_{k=2}^{\infty}\binom{\frac{1}{2}+\frac{1}{m}}{k}\left(\frac{\lambda_{1,n}}{\lambda_{0,n}}\right)^k + \frac{2m}{m+2}\sum_{j=1}^{\lfloor\frac{m}{2}+1\rfloor} d_{m,j}(a)\lambda_{0,n}^{-\frac{j}{m}}\left(\sum_{k=1}^{\infty}\binom{\frac{1}{2}+\frac{1-j}{m}}{k}\left(\frac{\lambda_{1,n}}{\lambda_{0,n}}\right)^k\right) + o\big(\lambda_{0,n}^{-\frac{1}{2}-\frac{1}{m}}\big)$$
$$= -\frac{2m}{m+2}\sum_{j=1}^{\lfloor\frac{m}{2}+1\rfloor} d_{m,j}(a)\lambda_{0,n}^{-\frac{j}{m}}. \tag{38}$$

Thus, one concludes $\frac{\lambda_{1,n}}{\lambda_{0,n}} = \lambda_{2,n} + \lambda_{3,n}$, where

$$\lambda_{2,n} = -\frac{2m}{m+2}d_{m,1}(a)\lambda_{0,n}^{-\frac{1}{m}} \qquad \text{and} \qquad \lambda_{3,n} = o\big(\lambda_{0,n}^{-\frac{1}{m}}\big). \tag{39}$$

Next, from (39) along with (38), we have

$$\lambda_{2,n} + \lambda_{3,n} + \frac{2m}{m+2}\sum_{k=2}^{\infty}\binom{\frac{1}{2}+\frac{1}{m}}{k}(\lambda_{2,n}+\lambda_{3,n})^k + \frac{2m}{m+2}\sum_{j=1}^{\lfloor\frac{m}{2}+1\rfloor} d_{m,j}(a)\lambda_{0,n}^{-\frac{j}{m}}\left(\sum_{k=1}^{\infty}\binom{\frac{1}{2}+\frac{1-j}{m}}{k}(\lambda_{2,n}+\lambda_{3,n})^k\right) + o(\lambda_{0,n}^{-\frac{1}{2}-\frac{1}{m}})$$
$$= -\frac{2m}{m+2}\sum_{j=1}^{\lfloor\frac{m}{2}+1\rfloor} d_{m,j}(a)\lambda_{0,n}^{-\frac{j}{m}}. \tag{40}$$

Thus,

$$\lambda_{3,n} + \frac{2m}{m+2}\sum_{k=2}^{\infty}\binom{\frac{1}{2}+\frac{1}{m}}{k}\sum_{\ell=0}^{k}\binom{k}{\ell}\lambda_{2,n}^{\ell}\lambda_{3,n}^{k-\ell} + \frac{2m}{m+2}\sum_{j=1}^{\lfloor\frac{m}{2}+1\rfloor} d_{m,j}(a)\lambda_{0,n}^{-\frac{j}{m}}\left(\sum_{k=1}^{\infty}\binom{\frac{1}{2}+\frac{1-j}{m}}{k}\sum_{\ell=0}^{k}\binom{k}{\ell}\lambda_{2,n}^{\ell}\lambda_{3,n}^{k-\ell}\right) + o\big(\lambda_{0,n}^{-\frac{1}{2}-\frac{1}{m}}\big)$$
$$= -\frac{2m}{m+2}\sum_{j=2}^{\lfloor\frac{m}{2}+1\rfloor} d_{m,j}(a)\lambda_{0,n}^{-\frac{j}{m}}, \tag{41}$$

and hence

$$\lambda_{3,n} + \frac{2m}{m+2}\sum_{k=2}^{\infty}\binom{\frac{1}{2}+\frac{1}{m}}{k}\sum_{\ell=0}^{k-1}\binom{k}{\ell}\lambda_{2,n}^{\ell}\lambda_{3,n}^{k-\ell} + \frac{2m}{m+2}\sum_{j=1}^{\lfloor\frac{m}{2}+1\rfloor} d_{m,j}(a)\lambda_{0,n}^{-\frac{j}{m}}\left(\sum_{k=1}^{\infty}\binom{\frac{1}{2}+\frac{1-j}{m}}{k}\sum_{\ell=0}^{k-1}\binom{k}{\ell}\lambda_{2,n}^{\ell}\lambda_{3,n}^{k-\ell}\right) + o\big(\lambda_{0,n}^{-\frac{1}{2}-\frac{1}{m}}\big)$$

$$= -\frac{2m}{m+2}\sum_{j=2}^{\lfloor \frac{m}{2}+1\rfloor} d_{m,j}(a)\lambda_{0,n}^{-\frac{j}{m}} - \frac{2m}{m+2}\sum_{k=2}^{\infty}\binom{\frac{1}{2}+\frac{1}{m}}{k}\lambda_{2,n}^{k}$$
$$- \frac{2m}{m+2}\sum_{j=1}^{\lfloor \frac{m}{2}+1\rfloor} d_{m,j}(a)\lambda_{0,n}^{-\frac{j}{m}}\left(\sum_{k=1}^{\infty}\binom{\frac{1}{2}+\frac{1-j}{m}}{k}\lambda_{2,n}^{k}\right). \qquad (42)$$

So we choose

$$\lambda_{3,n} = \lambda_{4,n} + \lambda_{5,n}, \qquad (43)$$

where

$$\lambda_{4,n} = -\frac{2m}{m+2}d_{m,2}(a)\lambda_{0,n}^{-\frac{2}{m}} - \frac{2m}{m+2}\binom{\frac{1}{2}+\frac{1}{m}}{2}\lambda_{2,n}^{2} - \frac{m d_{m,1}(a)}{m+2}\lambda_{0,n}^{-\frac{1}{m}}\lambda_{2,n}$$
$$= \left(-\frac{2m}{m+2}d_{m,2}(a) + \left(\frac{2m^2}{(m+2)^2} - \left(\frac{2m}{m+2}\right)^3\binom{\frac{1}{2}+\frac{1}{m}}{2}\right)d_{m,1}(a)^2\right)\lambda_{0,n}^{-\frac{2}{m}},$$

$$\lambda_{5,n} = o\left(\lambda_{0,n}^{-\frac{2}{m}}\right).$$

Next, we replace $\lambda_{3,n}$ in (42) by (43). Upon iterating this process we get

$$\lambda_n = \lambda_{0,n} + \lambda_{1,n} = \lambda_{0,n}\left(1 + \frac{\lambda_{1,n}}{\lambda_{0,n}}\right)$$
$$= \lambda_{0,n}(1 + \lambda_{2,n} + \lambda_{3,n})$$
$$= \lambda_{0,n}(1 + \lambda_{2,n} + \lambda_{4,n} + \lambda_{5,n})$$
$$\cdots$$
$$= \lambda_{0,n}\left(1 + \sum_{\ell=1}^{\lfloor \frac{m}{2}+1\rfloor} e_{\ell}(a)\lambda_{0,n}^{-\frac{\ell}{m}} + o\left(\lambda_{0,n}^{-\frac{1}{2}-\frac{1}{m}}\right)\right), \qquad (44)$$

as $n \to +\infty$, that is, (5).

Suppose that $m = 3$. For this case we will use the asymptotic expansion in theorem 14 that is valid in (27). Similarly to what we did for the case $m \geqslant 4$, if $C(a,\lambda) = 0$, then from the asymptotic expansion in theorem 14, we have

$$[1 + o(1)]\exp[L(G^4(a), \omega^{-2}\lambda) + L(G^2(a), \omega^{-1}\lambda)] = -\mathrm{i}\omega^{\frac{15}{4}}.$$

Thus, since $L(a,\lambda) = K_{3,0}(a)\lambda^{\frac{5}{6}} + K_{3,1}(a)\lambda^{\frac{3}{6}} + K_{3,2}(a)\lambda^{\frac{1}{6}} + o(1)$, we have

$$L(G^4(a), \omega^{-2}\lambda) + L(G^2(a), \omega^{-1}\lambda) + o(1)$$
$$= K_{3,0}(G^4(a))(\omega^{-2}\lambda)^{\frac{5}{6}} + K_{3,1}(G^4(a))(\omega^{-2}\lambda)^{\frac{3}{6}} + K_{3,2}(G^4(a))(\omega^{-2}\lambda)^{\frac{1}{6}}$$
$$+ K_{3,0}(G^2(a))(\omega^{-1}\lambda)^{\frac{5}{6}} + K_{3,1}(G^2(a))(\omega^{-1}\lambda)^{\frac{3}{6}} + K_{3,2}(G^2(a))(\omega^{-1}\lambda)^{\frac{1}{6}} + o(1)$$
$$= K_{3,0}\left(\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}} + \mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}\right)\lambda^{\frac{5}{6}} + c_{3,1}(a)\lambda^{\frac{3}{6}} + c_{3,2}(a)\lambda^{\frac{1}{6}} + o(1)$$
$$= -2\mathrm{i}K_{3,0}\sin\left(\frac{2\pi}{3}\right)\lambda^{\frac{5}{6}} + c_{3,1}(a)\lambda^{\frac{3}{6}} + c_{3,2}(a)\lambda^{\frac{1}{6}} + o(1).$$

So the continuous function $\lambda \mapsto L(G^4(a), \omega^{-2}\lambda) + L(G^2(a), \omega^{-1}\lambda) + o(1)$ maps a neighbourhood of the positive real axis near infinity onto a neighbourhood of the negative imaginary axis near infinity. Hence, there exists a sequence of λ_n near the positive real axis such that for all large enough positive integers n,

$$-2\mathrm{i}K_{3,0}\sin\left(\frac{2\pi}{3}\right)\lambda_n^{\frac{5}{6}} + c_{3,1}(a)\lambda_n^{\frac{3}{6}} + c_{3,2}(a)\lambda_n^{\frac{1}{6}} + o(1) = \ln\left(-\mathrm{i}\omega^{\frac{15}{4}}\right) = (\pi - 2(n+1)\pi)\mathrm{i}.$$

From this result one concludes that the asymptotic expansion (5) holds for $m = 3$ as well similarly to the proof for the case $m \geqslant 4$. □

6. Proof of theorem 3

First, note from (5) that $\arg(\lambda_n) \to 0$ as $n \to +\infty$.

Next, we have

$$\begin{aligned}
\lambda_{0,n+1} &= \left(\frac{(2n+3)\pi}{2K_{m,0}\sin\left(\frac{2\pi}{m}\right)}\right)^{\frac{2m}{m+2}} \\
&= \left(\frac{(2n+1)\pi}{2K_{m,0}\sin\left(\frac{2\pi}{m}\right)} + \frac{2\pi}{2K_{m,0}\sin\left(\frac{2\pi}{m}\right)}\right)^{\frac{2m}{m+2}} \\
&= \lambda_{0,n}\left(1 + \frac{2}{2n+1}\right)^{\frac{2m}{m+2}} \\
&= \lambda_{0,n}\left(1 + \frac{2m}{m+2}\frac{2}{2n+1} + O\left(\frac{1}{n^2}\right)\right) \\
&= \lambda_{0,n} + \frac{2m\pi}{(m+2)K_{m,0}\sin\left(\frac{2\pi}{m}\right)}\lambda_{0,n}^{1-\frac{1}{2}-\frac{1}{m}} + o\left(\lambda_{0,n}^{\frac{1}{2}-\frac{1}{m}}\right).
\end{aligned} \tag{45}$$

Thus,

$$\lambda_{n+1} - \lambda_n \underset{n\to+\infty}{=} \frac{2m\pi}{(m+2)K_{m,0}\sin\left(\frac{2\pi}{m}\right)}\lambda_{0,n}^{\frac{1}{2}-\frac{1}{m}} + o\left(\lambda_{0,n}^{\frac{1}{2}-\frac{1}{m}}\right),$$

and hence $|\lambda_{n+1} - \lambda_n| \to \infty$ and $\arg(\lambda_{n+1} - \lambda_n) \to 0$ as $n \to +\infty$. Since $\arg(\lambda_n) \to 0$ (and $\arg(\lambda_{n+1}) \to 0$) as $n \to +\infty$, there exists $N \in \mathbb{N}$ such that $|\lambda_n| < |\lambda_{n+1}|$ if $n \geqslant N$.

Remark. Here we will show that if $a \in \mathbb{R}^{m-1}$, then $e_\ell(a) \in \mathbb{R}$ for all $1 \leqslant \ell \leqslant \frac{m}{2} + 1$ with $e_\ell(a)$ defined in (44).

From (14) one can see that $\overline{K_{m,j}(G^{-1}(\overline{a}))} = K_{m,j}(G(a))$. Next, suppose that $a \in \mathbb{R}^{m-1}$. If $m \geqslant 4$, then from (34),

$$\begin{aligned}
\mathrm{i}c_{m,j}(a) &= \mathrm{i}\left(K_{m,j}(G(a))(\omega^2)^{\frac{1}{2}+\frac{1-j}{m}} - K_{m,j}(G^{-1}(a))(\omega^{-2})^{\frac{1}{2}+\frac{1-j}{m}}\right) \\
&= \mathrm{i}\left(K_{m,j}(G(a))(\omega^2)^{\frac{1}{2}+\frac{1-j}{m}} - \overline{K_{m,j}(G(a))(\omega^2)^{\frac{1}{2}+\frac{1-j}{m}}}\right) \in \mathbb{R}, \qquad 1 \leqslant j \leqslant \frac{m}{2} + 1.
\end{aligned}$$

So by (36), $d_{m,j}(a) \in \mathbb{R}$ for all $1 \leqslant j \leqslant \frac{m}{2} + 1$, and hence by (44), $e_\ell(a) \in \mathbb{R}$ for all $1 \leqslant \ell \leqslant \frac{m}{2} + 1$.

If $m = 3$, then one can show $e_\ell(a) \in \mathbb{R}$ for $\ell = 1, 2$, using the formulae at the end of the proof of theorem 2.

Acknowledgments

The author thanks Fritz Gesztesy and Richard Laugesen for critical reading of this manuscript and suggestions.

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